Table 1 Summary of results for four-mass oscillator

	Control effort reduction	Output cost reduction	C_1 norm	C_2 norm	Number of iterations
Spectral decomposition	5128%	21%	2.8×10^{-3}	4.4×10^{-2}	220

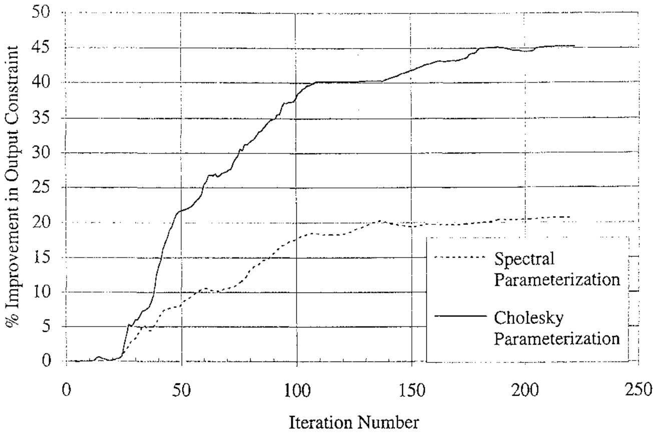

Fig. 2 Percentage reduction in output constraint vs iteration number for four-mass problem.

Using our parameterization of the controller, there are 210 design variables and 178 constraints. The results are shown in Figs. 1 and 2. Figure 1 plots the normalized control effort vs iteration number and Fig. 2 plots the percentage reduction in the output singular value.

The optimizer quickly found a design that satisfied the requested improvement of 10%. It achieved this in 57 iterations where one iteration is one unconstrained minimization. However, we allowed the algorithm to continue to search beyond the requested performance constraint. This resulted in a significant improvement in both the output constraint and the control effort, as summarized in Table 1.

Conclusions

A new parameterization of controllers for application to multiobjective optimizations has been developed and investigated. In this method, the spectral decomposition of the closed-loop covariance matrix is used as the design variables for the controller. This parameterization allows stability to be expressed as a simple side constraint on the design parameters, and performance and robustness constraints can be expressed as differentiable functions of the design parameters without the need for an intervening Lyapunov or eigenproblem solution.

Acknowledgments

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Optimization for Efficient Structure-Control Systems

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Introduction

REFERENCE 1 introduced the concept of power efficiency of structure-control systems (SCSs) and presented analyses of several linear quadratic regulator designs on the basis of their efficiencies. Encouraged by the results of Ref. 1, Ref. 2 introduced an efficiency modal analysis of an SCS. The efficiency modal analysis leads to identification of controller efficiency modes distinct from the structural natural modes. Two types of efficiency are defined in Ref. 1 for the SCS: global efficiency and the relative model efficiency. In this Note the focus is on the relative model efficiency. The model efficiency compares the control power expended on a reduced-order model to the total power expenditure on a full-order truth model (TM) via spatially discrete control inputs. In case of finite element models (FEMs) of structural systems the full-order system is the high-dimensional first-cut model of the system known as the Nth-order TM, where N is the total FEM structural degrees of freedom. In the case of distributed-parameter partial differential equation formulation, the full-order model is the ∞-dimensional system. A key feature in controlling a reduced-order model of a high-dimensional (or an ∞-dimensional distributed-parameter) structural dynamic system must be to achieve high power efficiency of the control system while satisfying the control objectives and/or constraints. Formally, this can be achieved by designing the control system and structural parameters simultaneously within an optimization framework. The subject of this Note is to present such a design procedure. Further details on the material presented here can be found in Refs. 3 and 4.

Efficiency Analysis for Structure-Control System

Consider an Nth-order FEM TM of the structural system

$$M\ddot{q} + Kq = DF(t) \tag{1}$$

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where M, K, and D are the mass, stiffness, and input influence matrices; $q(t)$ is the N-vector of nodal displacements; and $F(t)$ is the m-vector of point inputs. To control the structure described by Eq. (1), reduced-order modal state-space equations are considered,

$$\dot{x} = Ax + BF(t), \qquad x = \left[\eta_c^T \dot{\eta}_c\right]^T \tag{2}$$

where η_c are the $n < N$ structural modes controlled. The modal state-space system of Eqs. (2) is the reduced $(2n)$th-order control design model.

Because of any arbitrary input $F(t)$, the total control power S^R expended by the inputs on the actual full-order TM (1) and the portion S_C^M of this total expended power on the actual physical system that is projected onto the reduced-order dynamic system represented by Eqs. (2) are given by

$$S^R = \int F^T D^T M^{-1} DF \, \mathrm{d}t, \qquad S_C^M = \int F^T B^T BF \, \mathrm{d}t \tag{3}$$

We refer to S^R as the real (total) control power expended and S_C^M as the modal control power expended on the modal control design model. From Ref. 1, the relative model power efficiency is defined as $e\% = (S_C^M/S^R) \times 100$. If the input $F(t)$ on the physical system has the functional form of the state feedback of the reduced-control design model (2) as $F(t) = Gx$ where G is a stabilizing constant control feedback gain matrix of dimension $m \times 2n$, the model efficiency becomes

$$e = \frac{x_0^T P_C^M x_0}{x_0^T P^R x_0}, \qquad x_0 = x(t_0) \tag{4}$$

where P^R and P_C^M are symmetric positive-definite matrices referred to as real and modal control power matrices, respectively. They are the solutions of the Lyapunov equations associated with the closed-loop control system

$$A_{\mathrm{cl}}^T P^R + P^R A_{\mathrm{cl}} + G^T D^T M^{-1} DG = 0,$$

$$A_{\mathrm{cl}}^T P_C^M + P_C^M A_{\mathrm{cl}} + G^T B^T BG = 0, \qquad A_{\mathrm{cl}} = A + BG \tag{5}$$

Both power matrices are $(2n)$th-order; they are computed based on the reduced control design model. However, note that the real power matrix P^R inherently involves the TM through the appearance of the mass matrix M.

Next, we consider the efficiency eigenvalue problem associated with the power matrices (Ref. 2)

$$P_C^M t_i = \lambda_i^e P^R t_i, \qquad i = 1, 2, \ldots, 2n \tag{6}$$

where λ_i^e and t_i are defined as the ith characteristic efficiency and the ith controller efficiency mode, respectively. The eigenvectors t_i are also referred to as the principal controller efficiency modes, and they are normalized with respect to the power matrices. For any arbitrary vector (initial disturbance state) x_0, the value of the efficiency is bracketed by $\lambda_1^e \leq e \leq \lambda_{2n}^e \leq 1$ where the upper bound of 1 follows from the property that relative model efficiency cannot be greater than 100%. The term λ_1^e is referred to as the fundamental efficiency of the SCS. It is the minimum efficiency achievable by the SCS regardless of the initial state x_0. Again, since the model efficiency is stationary around an eigenvalue λ_i^e, it follows that if the initial disturbance $x_0 = t_i$, that is, if it matches the ith controller efficiency mode exactly, the efficiency will be exactly λ_i^e. We refer the reader to Ref. 2 for more discussions on the efficiency modes.

Optimization Problem Formulation

A high efficiency of any given reduced-order control design model implies that there is a lower fraction of control power spilled over to the truncated dynamics and hence minimized residual interaction with the design model. Furthermore, by definition, a high efficiency simply means more efficient use of resources available, which is a commonsense engineering design principle. We can then pose a structure-control optimization problem that incorporates the control system power efficiency.

Optimization problem: Objective → Minimize the total structural weight subject to constraints on the reduced-order control design model ($i \in \{1, 2, \ldots, n\}$):

$$\min \zeta_i \geq \zeta_i^* \tag{7a}$$

$$\omega_i \geq \omega_i^* \tag{7b}$$

$$e\% \geq e^*\% \tag{7c}$$

The control system design performance index (CDPI) is given by

$$\mathrm{CDPI} = \mathrm{Min}\ \frac{1}{2}\int_0^\infty (x^T \bar{Q} x + F^T \bar{R} F)\, \mathrm{d}t$$

$$\bar{Q} = \delta Q \geq 0, \qquad \bar{R} = \gamma R > 0 \tag{8}$$

where Q and R are properly specified constant matrices and δ and γ are the control system design variables. The system design variables are {Structural cross sections, γ, δ}.

Alternately, the efficiency constraint (7c) can be substituted by a constraint on the fundamental efficiency

$$\lambda_1^e > e^* \tag{7d}$$

guaranteeing a lower bound on the model efficiency regardless of initial disturbances where sensitivity of λ_1^e depends only on the system matrices via the efficiency eigenvalue problem (6). Hence, we solve the optimization problem subject to the constraints (7a), (7b), and (7d).

In Eqs. (7), i denotes a chosen set of modes from a set of n modes in the design space; ζ_i is the damping ratio of the controlled system and ω_i is the closed-loop frequency. An asterisk denotes minimum desirable constraint values. The novel feature of the problem posed here is the inclusion of the nondimensional SCS parameter, the power efficiency e in addition to the already too familiar nondimensional parameter, the damping ratio ζ. The constraint on ζ reflects a concern on the quality of response, whereas the constraint on e reflects a concern on the efficient use of the control power. An equally important feature of this optimization formulation is that the goodness of the reduced-order design model relative to the full-order TM is explicitly but intricately incorporated to the design via the introduction of the efficiency constraint.

The sensitivity expressions for the objective and constraint functions are given in Refs. 3 and 4 and the references listed therein.

Illustrative Examples

The ACOSS-FOUR (Active Control of Space Structures) tetrahedral model shown in Fig. 1 of Ref. 2 was used to design a minimum-weight structure with constraints on the closed-loop eigenvalues and the fundamental efficiency. This structure has 12 degrees of freedom ($N = 12$) and four masses of two units each attached at nodes 1–4. Node 1 is the vertex of the structure. Nodes 2–4 form the base and each is anchored by bipods. Thus there are 12 structural members. The dimensions and the elastic properties of the structure are specified in consistent nondimensional units in Ref. 7 cited in Ref. 4. Six collocated actuators and sensors are in six bipods. The control approach used is the linear quadratic regulator with steady-state gain feedback via minimizing the control design performance index [Eqs. (8)]. In Eqs. (8), the weighing matrices Q and R for the state and control variables were assumed to be equal to the identity matrices and the parameters δ and γ were used as design variables along with the 12 structural member cross-sectional areas.

The nominal initial design is denoted as design A. This initial design weighs 43.69 units.

The constraints imposed on the optimum designs were $\omega_1 \geq 1.1\omega_1$ (initial), $\omega_2 \geq 1.1\omega_2$ (initial), $\zeta_1 \geq 1.5\zeta_1$ (initial), $\lambda_1 = e_{\min} \geq 1.75\lambda_1$ (initial) $= 1.75 e_{\min}$ (initial). We should note that the designation with subscript 1 in the efficiency constraint refers to the first efficiency mode or first principal controller mode, not to the first structural mode. The NEWSUMT-A software based on the

Table 1 Optimum designs

Constraints	Design A, initial design, $n = 8, 6$	Design B, $n = 8$ modes, $m = 6$ inputs	Design C, $n = 6$ modes, $m = 3$ inputs
ω_1	1.295	1.425	1.425
ω_2	1.596	1.757	1.757
ζ_1	0.056, 0.031	0.290	0.130
λ_1, %	40.7, 52.7	71.2	92.1
S^R	4.88, 17.88	89.00	26.02
S_C^M	3.00, 9.59	78.91	24.74
e, %	61.5, 53.6	88.6	95.1
Weight	43.69	21.97	26.79

extended interior penalty function method with Newton's method of unconstrained-minimization was used to obtain optimum designs. Two optimization problems were solved, each with a different reduced-order control design model and a different input configuration. These designs were denoted as designs B and C. Design B used the first eight natural structural modes in the reduced-order control design model ($n = 8$) and six inputs ($m = 6$) located on the six bipods of the structure. Design C used the first six natural structural modes in the reduced-order control design model ($n = 6$) with two inputs ($m = 2$) located on the two bipods attached to node 2.

The results of optimizations are given in Table 1, which includes values obtained for the constrained quantities ω_1, ω_2, λ_1^e, and ζ_1 and the objective functions, weights of the structures. In addition, the resulting real control powers expended, S^R, and the amount of this power that was absorbed by the reduced-order design models, S_C^M, and the respective model efficiencies, $e\%$, are listed in Table 1. As the initial disturbance state x_0, a unit displacement in the x-direction at node 2 was assumed. Note that the initial disturbance affects only the value of model efficiency e, not the value of the fundamental efficiency λ_1^e.

For all designs, A–C, we denote the set of design variables which are the 12-member cross sections and the control weighting parameters δ and γ by $\{DV\}$, the set of squared structural frequencies by $\{\omega^2\}$, the set of damping ratios by $\{\zeta\}$, and the set of characteristic efficiencies by $\{\lambda^e\}$. The following numerical results were obtained:

$$\{DV\}_A = \{10^3 \quad 10^3 \quad 10^2 \quad 10^2 \quad 10^3 \quad 10^3 \quad 10^2 \quad 10^2 \quad 10^2 \quad 10^2 \quad 10^2 \quad 10^2 \quad 1.00 \quad 1.00\}$$

$$\{DV\}_B = \{246.9 \quad 403.6 \quad 175.4 \quad 257.2 \quad 228.2 \quad 253.9 \quad 54.2 \quad 226.1 \quad 224.6 \quad 528.2 \quad 604.9 \quad 597.4 \quad 4.24 \quad 0.24\}$$

$$\{DV\}_C = \{126.5 \quad 280.7 \quad 575.5 \quad 576.2 \quad 407.9 \quad 277.1 \quad 84.7 \quad 68.2 \quad 547.8 \quad 171.8 \quad 416.2 \quad 270.8 \quad 2.45 \quad 0.41\}$$

$$\{\omega^2\}_A = \{1.68 \quad 2.55 \quad 7.31 \quad 7.52 \quad 9.98 \quad 16.06 \quad 20.01 \quad 20.17 \quad 66.24 \quad 77.46 \quad 97.42 \quad 151.30\}$$

$$\{\omega^2\}_B = \{2.11 \quad 3.27 \quad 7.83 \quad 11.17 \quad 17.34 \quad 22.80 \quad 44.61 \quad 50.40 \quad 50.52 \quad 96.96 \quad 107.40 \quad 110.70\}$$

$$\{\omega^2\}_C = \{2.06 \quad 3.15 \quad 8.43 \quad 13.85 \quad 19.27 \quad 24.17 \quad 24.43 \quad 43.32 \quad 55.84 \quad 70.42 \quad 92.49 \quad 112.86\}$$

$$\{\zeta\}_A^{n=8} = 10^{-3}\{56 \quad 67 \quad 74 \quad 81 \quad 85 \quad 87 \quad 76 \quad 72\},$$

$$\{\zeta\}_A^{n=6} = 10^{-3}\{31 \quad 34 \quad 9 \quad 63 \quad 77 \quad 49\}$$

$$\{\zeta\}_B = 10^{-3}\{290 \quad 107 \quad 335 \quad 106 \quad 100 \quad 189 \quad 205 \quad 196\}$$

$$\{\zeta\}_C = 10^{-3}\{130 \quad 171 \quad 121 \quad 120 \quad 160 \quad 124\}$$

$$\{\lambda^e\}_A^{n=8} = \%\{40.77 \; - 99.98\}, \qquad \{\lambda^e\}_A^{n=6} = \%\{52.65 \; - 59.95\}$$

$$\{\lambda^e\}_B = \%\{71.24 \; - 99.98\}, \qquad \{\lambda^e\}_C = \%\{92.14 \; - 97.85\}$$

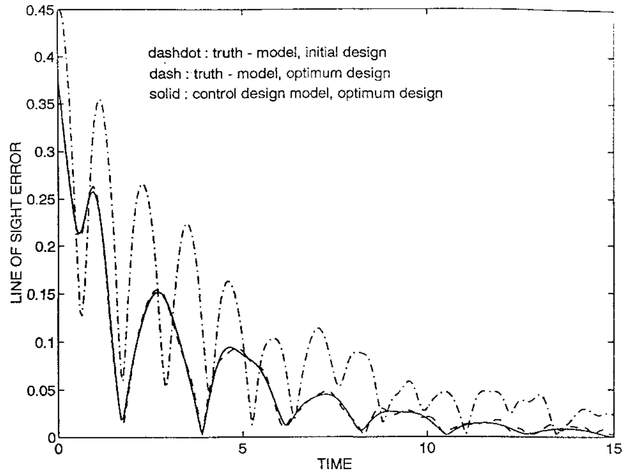

Fig. 1 Line-of-sight error responses for initial design A and optimum design B.

The subscripts A, B, and C denote the particular designs and n denotes the number of design modes. For brevity, for the characteristic efficiency spectrum we listed the minimum and maximum bracket values. For a qualitative discussion of the optimization results we refer the reader to Ref. 4, which is the original version of this Note.

Figure 1 shows the line-of-sight error responses at the vertex (node 1) of the 12-mode TMs of the initial design (design A, dashdot curve) and the optimum design (design B, dashed cruve) for the control design model of eight lowest natural modes and six inputs. Design B has an efficiency of 88.6% vs the 61.5% efficiency of the initial design A. Note that due to the high efficiency achieved in the optimum design B, the TM response and the control design model response (solid curve) are almost identical. Due to space limitations we do not show simulation results for design C.

The relative model efficiency e is a figure of merit that can also be used to ascertain the quality of response of the TM of a controlled structure based on the study and simulation of a reduced-order control design model without any need for simulation of the TM. It is known that the mean-square distributed response of truncated dynamics is inversely proportional to the fourth power of the truncated natural frequencies and directly proportional to the time-weighted control power spilled over to the truncated dynamics, which is reflected in the power inefficiency of the system. If the truncated frequencies are high frequencies and a high system efficiency is realized, then one would hardly expect any degradation of the response of the reduced-order control design model due to excitation of truncated dynamics. In contrast, if the truncated frequencies are of low natural frequencies, then the inefficiency figure will further be magnified when it is correlated to the system response. In case of such low frequencies in the truncated dynamics it becomes even of more concern to obtain very high power efficiencies. In any case, even if the effect of truncated dynamics on the response is ascertained to be insignificant, the power efficiency of the SCS in achieving this and any other control objectives must still be considered.

Finally, some remarks are in order as to the choice of different input configurations for the two optimum designs B and C. As discussed in Ref. 1, efficiency can also be used as an indicator of the effects of changes in the input configuration and the design model order. It is illustrated in Ref. 1 that for the initial design A with the six-mode design model inclusion of inputs 3–6 degrades the efficiency of the system, whereas their inclusion improves the efficiency of the system for the eight-mode design model. Thus, for the optimization problems formulated with the objective of improving the efficiencies of the eight- and six-mode reduced-order designs, from the study of efficiencies of the initial design A, the input configurations were chosen with six inputs and the first two inputs, respectively, culminating in satisfaction of our objectives for both designs.

Conclusions

Incorporation of the efficiency concept as a norm of the SCS design and analysis enhances the overall quality of the system. Structure-control system efficiency is a physically based nondimensional parameter indicating the degree of usefulness of a fundamental quantity in the design and analysis of many engineering disciplines, namely, the power. This Note demonstrates that a focus on the system control power efficiency does not curtail the designer's ability in monitoring other important quantities of the overall design; on the contrary, it brings in an added, but necessary, dimension to the SCS that is a time-tested proven concept in engineering design. The improvement of efficiency, in the least, simply makes better use of available control power since it results in reduced power interaction with the unmodeled dynamics. Furthermore, this reduction is not merely qualitative but it is quantified via efficiency. More importantly, all efficiency computations only involve the reduced-order control design model while extracting information about the behavior of the truth-model of the system.

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Polynomial Interpolation Between Input Samples for Continuous-Time Simulation

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Introduction

EFFICIENT techniques for computing the continuous-time response of linear time-invariant (LTI) systems using polynomial interpolation between input samples are presented in this Note. Usually the response of an LTI system is computed by discretizing the continuous-time equations with a zero-order hold approximation and performing discrete-time simulation with uniformly spaced input samples. This approach gives fairly accurate results when the time step T is small; in particular, for $T < \pi/|\lambda_{\max}|$, where $\lambda_{\max}$ is the eigenvalue of A with largest absolute value.[1] Linear interpolation between control samples can be achieved by appending integrators at each input channel of the given system, discretizing this combined system with zero-order hold, and applying the derivative of the input to this system,[2] though this approach increases the order of the system. This Note develops techniques to evaluate the response of an LTI system in state-space form by interpolating between the input samples with polynomials of arbitrary order. Further, efficient algorithms are presented for computation of certain matrix integrals required for implementation of these techniques. A numerical example is presented to demonstrate the benefits of these techniques.

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Response with Polynomial Interpolation

Consider a linear time-invariant system $\dot{x} = Ax + Bu$, $y = Cx + Du$, where A, B, C, D is a realization of the system. Assume that the input is specified at uniformly spaced time steps T, that is, $u[kT]$ for $k = 0, 1, 2, \ldots$ is specified. From the state of the system, $x[kT]$, at the instant kT, the state at the next instant, $x[(k+1)T]$, is obtained from

$$x[(k+1)T] = e^{AT}x[kT] + \int_0^T e^{A(T-\tau)}Bu(\tau + kT)\,\mathrm{d}\tau \tag{1}$$

and the output becomes $y[kT] = Cx[kT] + Du[kT]$. The expressions for evaluating the state response with polynomial approximations for $u(t)$ in the intersample interval, $kT \le t \le (k+1)T$, are developed in this section.

With zero-order polynomial approximation, the input is constant between two samples, that is, $u(t) = a_0[kT]$ for $kT \le t \le (k+1)T$. The constant coefficients $a_0[kT]$ for $k = 0, 1, 2, \ldots$ are determined by the interpolation scheme used, usually $a_0[kT] = u[kT]$. Defining

$$H_0(t) = \int_0^t e^{A(t-\tau)}\,\mathrm{d}\tau$$

the state at instant $(k+1)T$ for zero-order hold becomes

$$x[(k+1)T] = e^{AT}x[kT] + H_0(T)Ba_0[kT] \tag{2}$$

These equations correspond to the well-known zero-order hold discretization of continuous-time system equations.

First-order polynomial approximation of the intersample behavior of the input is expressed as $u(t) = a_0[kT] + a_1[kT](t - kT)$ for $kT \le t \le (k+1)T$. For linear interpolation between $u[kT]$ and $u[(k+1)T]$, $a_0[kT] = u[kT]$ and $a_1[kT] = \{u[(k+1)T] - u[kT]\}/T$. Defining

$$H_1(t) = \int_0^t \tau e^{A(t-\tau)}\,\mathrm{d}\tau$$

the evolution of the state vector is expressed as

$$x[(k+1)T] = e^{AT}x[kT] + H_0(T)Ba_0[kT] + H_1(T)Ba_1[kT] \tag{3}$$

Approximation of the intersample behavior of the input by nth-order polynomials ($n \ge 2$) can be expressed as

$$u(t) = \sum_{i=0}^{n} a_i[kT](t - kT)^i, \qquad kT \le t \le (k+1)T \tag{4}$$

The coefficients $a_i[kT]$, $i = 0, 1, \ldots, n$, are selected to prescribe the desired intersample behavior between input samples. Using Eq. (1), the state $x[(k+1)T]$ becomes

$$x[(k+1)T] = e^{AT}x(kT) + \sum_{i=0}^{n} H_i(T)Ba_i[kT] \tag{5}$$

where

$$H_i(t) = \int_0^t \tau^i e^{A(t-\tau)}\mathrm{d}\tau$$

With these formulas, polynomials of arbitrary order can be used to interpolate between the input samples for computation of the continuous-time response of linear time-invariant systems.

Evaluating the Matrix Integrals

Efficient evaluation of the matrix integrals $H_i(t)$ is the critical issue in implementation of this approach for computing the response of a linear time-invariant system. This section presents an efficient algorithm for robust computation of these matrix integrals, based on Padé approximations for matrix exponentials. For simplicity, the following discussion is limited to computation of $H_i(t)$ for $i = 0, 1, 2$.